線形代数 (第 9 回) の解答

問題 9-1 の解答

「① が ② へ行き」, 「② が ④ へ行き」, 「③ が ① へ行き」, 「④ が ③ へ行く」という置換なので,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

問題 9-2 の解答

(1) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ と置くと,

$$(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(4) = 1, \qquad (\sigma\tau)(2) = \sigma(\tau(2)) = \sigma(3) = 3,$$
$$(\sigma\tau)(3) = \sigma(\tau(3)) = \sigma(2) = 4, \qquad (\sigma\tau)(4) = \sigma(\tau(4)) = \sigma(1) = 2.$$

よって

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

(2) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ と置く. $\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ より,

$$(\sigma\tau^{-1})(1) = \sigma(\tau^{-1}(1)) = \sigma(1) = 2, \qquad (\sigma\tau^{-1})(2) = \sigma(\tau^{-1}(2)) = \sigma(4) = 1,$$
$$(\sigma\tau^{-1})(3) = \sigma(\tau^{-1}(3)) = \sigma(2) = 3, \qquad (\sigma\tau^{-1})(4) = \sigma(\tau^{-1}(4)) = \sigma(3) = 4.$$

よって

$$\sigma\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

問題 9-3 の解答

(1) $\sigma = (1, 2, 3, 4)$, $\tau = (1, 2)$ と置くと,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

よって

$$(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(2) = 3, \qquad (\sigma\tau)(2) = \sigma(\tau(2)) = \sigma(1) = 2,$$

$$(\sigma\tau)(3) = \sigma(\tau(3)) = \sigma(3) = 4, \qquad (\sigma\tau)(4) = \sigma(\tau(4)) = \sigma(4) = 1.$$

従って

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(2) $\sigma = (1,3,4)$, $\tau = (1,2,3)$, $\rho = (1,2,4)$ と置くと,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

よって

$$\begin{aligned}
(\sigma\tau\rho)(1) &= \sigma(\tau(\rho(1))) = \sigma(\tau(2)) = \sigma(3) = 4, \\
(\sigma\tau\rho)(2) &= \sigma(\tau(\rho(2))) = \sigma(\tau(4)) = \sigma(4) = 1, \\
(\sigma\tau\rho)(3) &= \sigma(\tau(\rho(3))) = \sigma(\tau(3)) = \sigma(1) = 3, \\
(\sigma\tau\rho)(4) &= \sigma(\tau(\rho(4))) = \sigma(\tau(1)) = \sigma(2) = 2.
\end{aligned}$$

従って

$$\sigma\tau\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

問題 9-4 の解答

巡回部分は次の 3 つ.

$$\begin{aligned}
&1 \mapsto 10 \mapsto 5 \mapsto 2 \mapsto 3 \mapsto 9 \mapsto 1 \\
&4 \mapsto 7 \mapsto 4 \\
&6 \mapsto 8 \mapsto 6
\end{aligned}$$

よって

$$\sigma = (1,10,5,2,3,9)(4,7)(6,8).$$

問題 9-5 の解答

σ を巡回置換の積に分解すると,

$$\sigma = (1,5)(2,6,4)(3,7,9,8).$$

次に, それぞれの巡回置換を互換の積に分解すると,

$$(2,6,4) = (2,4)(2,6), \qquad (3,7,9,8) = (3,8)(3,9)(3,7).$$

よって, σ の互換の積への分解は

$$\sigma = (1,5)(2,4)(2,6)(3,8)(3,9)(3,7).$$

σ は 6 個の互換の積に分解されたので, $\mathrm{sgn}(\sigma) = (-1)^6 = 1$.