教養の微積 (16 回目) の解答

問題 16-1 の解答

(1) $y = 3x + 1$ とおく．$\frac{1}{3}dy = dx$ より，

$$\int (3x+1)^5 \, dx = \int \frac{y^5}{3} \, dy = \frac{y^6}{18} + C = \frac{1}{18}(3x+1)^6 + C.$$

(2) $y = x^2$ とおく．$\frac{1}{2}dy = xdx$ より，

$$\int xe^{-x^2} \, dx = \int \frac{1}{2} \, e^{-y} \, dy = -\frac{1}{2} \, e^{-y} + C = -\frac{1}{2} \, e^{-x^2} + C.$$

(3) $y = e^x + 1$ とおくと $dx = \frac{1}{y-1}dy$．よって

$$\int \frac{1}{e^x+1} \, dx = \int \frac{1}{y(y-1)} \, dy = \int \frac{1}{y-1} - \frac{1}{y} \, dy = \log|y-1| - \log|y| + C = x - \log(e^x+1) + C.$$

(4) $y = x + 1$ とおくと $dx = dy$．また $x : 0 \to 1$ のとき, $y : 1 \to 2$ となる．よって

$$\int_0^1 x\sqrt{x+1} \, dx = \int_1^2 (y-1)\sqrt{y} \, dy = \left[\frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right]_1^2 = \left(\frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{4(\sqrt{2}+1)}{15}.$$

(5) $x = \sqrt{2}\sin y$ と置くと，

$$dx = \frac{dx}{dy} \, dy = \sqrt{2}\cos y \, dy.$$

また $x : 0 \to 1$ のとき, $y : 0 \to \frac{\pi}{4}$ である．$\cos y \geqq 0$ より

$$\sqrt{2-x^2} = \sqrt{2}\sqrt{1-(\sin y)^2} = \sqrt{2}\cos y.$$

よって

$$\int_0^1 \sqrt{2-x^2} \, dx = 2\int_0^{\frac{\pi}{4}} (\cos y)^2 \, dy = \int_0^{\frac{\pi}{4}} \cos(2y) + 1 \, dy = \left[\frac{\sin(2y)}{2} + y\right]_0^{\frac{\pi}{4}} = \frac{2+\pi}{4}.$$

問題 16-2 の解答

(1) について．

$$\int \frac{2x}{x^2+3} \, dx = \int \frac{(x^2+3)'}{x^2+3} \, dx = \log(x^2+3) + C.$$

(2) について.

$$\int_0^1 \frac{e^x}{e^x+1}\,dx = \int_0^1 \frac{(e^x+1)'}{e^x+1}\,dx = \Big[\log(e^x+1)\Big]_0^1 = \log(e+1) - \log 2 = \log\left(\frac{e+1}{2}\right).$$

(3) について.

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{\cos x}\,dx = -\Big[\ \log(\cos x)\ \Big]_0^{\frac{\pi}{4}} = \log\sqrt{2}.$$

問題 16-3 の解答

(1) $\left(\sin\left(\frac{\theta}{2}\right)\right)^2 + \left(\cos\left(\frac{\theta}{2}\right)\right)^2 = 1$ より

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\left(\sin\left(\frac{\theta}{2}\right)\right)^2 + \left(\cos\left(\frac{\theta}{2}\right)\right)^2}$$

分母・分子を $\left(\cos\left(\frac{\theta}{2}\right)\right)^2$ で割ると

$$\sin\theta = \frac{2t}{1+t^2}.$$

また

$$\cos\theta = \left(\sin\left(\frac{\theta}{2}\right)\right)^2 - \left(\cos\left(\frac{\theta}{2}\right)\right)^2 = \frac{\left(\sin\left(\frac{\theta}{2}\right)\right)^2 - \left(\cos\left(\frac{\theta}{2}\right)\right)^2}{\left(\sin\left(\frac{\theta}{2}\right)\right)^2 + \left(\cos\left(\frac{\theta}{2}\right)\right)^2}$$

より, 分母・分子を $\left(\cos\left(\frac{\theta}{2}\right)\right)^2$ で割ると

$$\cos\theta = \frac{1-t^2}{1+t^2}.$$

次に

$$\frac{dt}{d\theta} = \frac{1}{2\left(\cos\left(\frac{\theta}{2}\right)\right)^2} = \frac{1}{2}\left\{\left(\tan\left(\frac{\theta}{2}\right)\right)^2 + 1\right\} = \frac{t^2+1}{2}.$$

よって

$$\frac{2dt}{1+t^2} = d\theta.$$

(2) について.

(i) $\theta : 0 \to \frac{\pi}{3}$ のとき, $t : 0 \to \frac{1}{\sqrt{3}}$ なので, (1) より

$$\int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{1-t^2}\,dt = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1-t} + \frac{1}{1+t}\,dt.$$

よって,

$$\int_0^{\frac{\pi}{3}} \frac{1}{\cos\theta}\,dx = \Big[-\log(1-t) + \log(1+t)\Big]_0^{\frac{1}{\sqrt{3}}} = \log(2+\sqrt{3}).$$

(ii) $\theta : 0 \to \frac{\pi}{2}$ のとき, $t : 0 \to 1$ なので, (1) より

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,dx = \int_0^1 \frac{2}{(1+t)^2}\,dt = \left[\frac{-2}{1+t}\right]_0^1 = 1.$$