教養の微積 (17 回目) の解答

問題 17-1 の解答

(1)

$$\int x\log x\ dx = \int \left(\frac{x^2}{2}\right)' \log x\ dx = \frac{x^2}{2}\log x - \int \frac{x}{2}\ dx = \frac{x^2}{2}\log x - \frac{x^2}{4} + C.$$

(2)

$$\begin{aligned}
\int x^2 e^x\ dx &= \int x^2 (e^x)'\ dx \\
&= x^2 e^x - \int 2xe^x\ dx \\
&= x^2 e^x - \int 2x(e^x)'\ dx \\
&= x^2 e^x - 2xe^x + \int 2e^x\ dx \\
&= x^2 e^x - 2xe^x + 2e^x + C.
\end{aligned}$$

(3)

$$\begin{aligned}
\int \arctan x\ dx &= \int (x)' \arctan x\ dx \\
&= x\arctan x - \int \frac{x}{1+x^2}\ dx \\
&= x\arctan x - \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}\ dx \\
&= x\arctan x - \frac{1}{2}\log(1+x^2) + C
\end{aligned}$$

問題 17-2 の解答

(1)

$$\begin{aligned}\int_1^e (2x+1)\log x\, dx &= \int_1^e \left(x^2+x\right)' \log x\, dx \\ &= \Big[(x^2+x)\log x\Big]_1^e - \int_1^e x+1\, dx \\ &= (e^2+e) - \left[\frac{x^2}{2}+x\right]_1^e \\ &= \frac{e^2+3}{2}.\end{aligned}$$

(2)

$$\begin{aligned}\int_0^\pi x^2\sin x\, dx &= \int_0^\pi x^2\ (-\cos x)'\, dx \\ &= \Big[-x^2\cos x\Big]_0^\pi + 2\int_0^\pi x\cos x\, dx \\ &= \pi^2 + 2\int_0^\pi x(\sin x)'\, dx \\ &= \pi^2 + 2\left\{\Big[x\sin x\Big]_0^\pi - \int_0^\pi \sin x\, dx\right\} \\ &= \pi^2 + 2\left\{0 + \Big[\cos x\Big]_0^\pi\right\} \\ &= \pi^2 - 4.\end{aligned}$$

(3)

$$\begin{aligned}\int_0^{\frac{1}{2}} \arcsin x\, dx &= \int_0^{\frac{1}{2}} (x)' \arcsin x\, dx \\ &= \Big[x\arcsin x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx \\ &= \frac{\pi}{12} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx\end{aligned}$$

後の積分を計算する．$y = 1-x^2$ と置くと，$\frac{dy}{dx} = -2x$ より

$$\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\, dx = \int_1^{\frac{3}{4}} \frac{-1}{2\sqrt{y}}\, dy = \int_{\frac{3}{4}}^1 \frac{1}{2\sqrt{y}}\, dy = \Big[\sqrt{y}\Big]_{\frac{3}{4}}^1 = 1 - \frac{\sqrt{3}}{2}.$$

以上より

$$\int_0^{\frac{1}{2}} \arcsin x\, dx = \frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}.$$

問題 17-3 の解答

(1)

$$I_n = \int_0^1 \frac{1}{(x^2+1)^n} \cdot (x)' \, dx$$

$$= \Big[\frac{x}{(x^2+1)^n}\Big]_0^1 + 2n \int_0^1 \frac{x^2}{(x^2+1)^{n+1}} \, dx$$

$$= \frac{1}{2^n} + 2n \int_0^1 \frac{(x^2+1)}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}} \, dx$$

$$= \frac{1}{2^n} + 2n(I_n - I_{n+1}).$$

よって

$$I_{n+1} = \frac{1}{n2^{n+1}} + \frac{(2n-1)}{2n} I_n.$$

(2) (1) より

$$I_3 = \frac{1}{16} + \frac{3}{4} \cdot I_2, \qquad I_2 = \frac{1}{4} + \frac{1}{2} \cdot I_1.$$

また

$$I_1 = \int_0^1 \frac{1}{x^2+1} \, dx = \Big[\arctan x\Big]_0^1 = \frac{\pi}{4}.$$

よって

$$I_3 = \frac{8+3\pi}{32}.$$

問題 17-4 の解答

(1)

$$(\cosh(x))^2 - (\sinh(x))^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= 1.$$

(2)

$$(\sinh(x))' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x).$$

(3) $(\sinh(x))' = \frac{e^x+e^{-x}}{2} > 0$ より，$\sinh(x)$ は $(-\infty,\infty)$ 上で単調増加でその値域は $(-\infty,\infty)$．従って $\sinh^{-1}(x)$ の定義域は $(-\infty,\infty)$．ここで，

$$
\begin{aligned}
y = \frac{e^x - e^{-x}}{2} &\iff (e^x)^2 - 2ye^x - 1 = 0 \\
&\iff e^x = y + \sqrt{y^2+1} \qquad (e^x > 0 \text{ に注意}) \\
&\iff x = \log(y + \sqrt{y^2+1}).
\end{aligned}
$$

以上より，

$$\sinh^{-1}(x) = \log(x + \sqrt{x^2+1}) \quad (-\infty < x < \infty).$$

(4) $x = \sinh(y)$ と置く．$\frac{dx}{dy} = \cosh(y)$ なので $dx = \cosh(y)dy$．(1) より

$$\sqrt{1+x^2} = \sqrt{1+(\sinh(y))^2} = \sqrt{\cosh(y)^2} = \cosh(y).$$

以上より

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \int dy = y + C.$$

ここで，

$$x = \sinh(y) \iff y = (\sinh)^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right).$$

従って

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \log\left(x + \sqrt{x^2+1}\right) + C.$$

問題 17-5 の解答

定理 17-2 (2) より

$$
\begin{aligned}
L(C) &= \int_0^1 \sqrt{1+(2x)^2}\,dx \\
&= 2\int_0^1 \sqrt{\frac{1}{4}+x^2}\,dx \\
&= \left[x\sqrt{x^2+\frac{1}{4}} + \frac{1}{4}\log\left(x + \sqrt{x^2+\frac{1}{4}}\right)\right]_0^1 \\
&= \frac{2\sqrt{5} + \log(2+\sqrt{5})}{4}.
\end{aligned}
$$